Question 1 **(4 marks)**

Find the value of c, where $c \in \mathbb{R}$, such that the curve defined by

$$y^2 + \frac{3e^{(x-1)}}{x-2} = c$$

has a gradient of 2 where $x = 1$.

$$2y\frac{dy}{dx} + \frac{3e^{(x-1)}(x-2) - 3e^{(x-1)}}{(x-2)^2} = 0$$ ✓ Implicit diff.

$x = 1$: $$2y\frac{dy}{dx} + \frac{3(-1)-3}{1} = 0$$ ✓ substitute $x=1$

$$2y\frac{dy}{dx} - 6 = 0$$

$$\frac{dy}{dx} = \frac{6}{2y}$$

$$2 = \frac{3}{y}$$

$$y = 1.5$$ ✓ find y

$$\therefore \quad c = 2.25 + \frac{3}{-1}$$

$$c = -0.75$$ ✓

Question 2 **(6 marks)**

A container of water is heated to boiling point (100 °C) and then placed in a room that has a constant temperature of 20 °C. After five minutes the temperature of the water is 80 °C.

(a) Use Newton's law of cooling

$$\frac{dT}{dt} = -k(T - 20)$$

where T °C is the temperature of the water at time t minutes after the water is placed in the room, to show that $e^{-5k} = \frac{3}{4}$. (3 marks)

$\int \frac{dT}{T-20} = -k\int dt$ ✓ Separate variables

$\ln(T-20) = -kt + c$

$T - 20 = e^{-kt+c}$

$t=0: \quad 80 = e^{c}$ ✓ $(0, 100)$

$\therefore \quad T - 20 = 80e^{-kt}$

$t=5: \quad 60 = 80e^{-5k}$

$\therefore \quad e^{-5k} = \frac{3}{4}$ ✓ Arrives at correct answer.

(b) Determine the temperature of the water 10 minutes after it is placed in the room. (3 marks)

$T = 20 + 80e^{-kt}$ ✓

$t=10: \quad T = 20 + 80e^{-10k}$ ✓ for $t=10$

$= 20 + 80\left(e^{-5k}\right)^2$

$= 20 + 80\left(\frac{3}{4}\right)^2$

$= 20 + 80 \times \frac{9}{16}$

$= 65$ ✓

Question 3 **(5 marks)**

The equation of motion for a particle moving in simple harmonic motion is given by

$$\frac{d^2x}{dt^2} = -k^2x$$

where k is a positive constant, x is the displacement of the particle and t is time.

Show that $v^2 = k^2(a^2 - x^2)$, where v is the velocity and a is the amplitude of the motion.

$$\frac{dv}{dt} = -k^2 x$$ ✓ $\frac{dv}{dt}$

$$\frac{dv}{dx}\frac{dx}{dt} = -k^2 x$$ ✓ chain rule

$$\frac{dv}{dx} v = -k^2 x$$

$$\int v\, dv = -k^2 \int x\, dx$$ ✓ separate variable

$$\frac{v^2}{2} = \frac{-kx^2}{2} + c$$ ✓ integrate

When $x = a$, $v = 0$:

$$0 = \frac{-k^2a^2}{2} + c$$

$$c = \frac{k^2a^2}{2}$$

$$\therefore \frac{v^2}{2} = \frac{-k^2x^2}{2} + \frac{k^2a^2}{2}$$

$$\therefore v^2 = k^2(a^2 - x^2)$$ as required. ✓ correct working out

End of questions

Question 4 **(5 marks)**

The logistic model for growth can be described algebraically by a differential equation of the form

$$\frac{dy}{dt} = ay - by^2, \quad \text{with } a > 0 \text{ and } b > 0.$$

Use calculus to show that

$$y = \frac{a}{b + ce^{-at}}, \quad \text{where } c \text{ is some constant}$$

satisfies the above-mentioned differential equation.

$y = a(b + ce^{-at})^{-1}$

$\Downarrow$

$\frac{dy}{dt} = -a(b + ce^{-at})^{-2}(-ace^{-at})$ ✓ Finds $\frac{dy}{dt}$

$= \frac{a^2 ce^{-at}}{(b + ce^{-at})^2}$ ✓ correct $\frac{dy}{dt}$

$= y^2 ce^{-at}$ ✓ subs. y

Also, $y = \frac{a}{b + ce^{-at}}$

$\Downarrow$

$b + ce^{-at} = \frac{a}{y}$

$\Downarrow$

$ce^{-at} = \frac{a}{y} - b$ ✓ solve for ce^{-at}

$\therefore \frac{dy}{dt} = y^2\left(\frac{a}{y} - b\right)$ ✓ substitute

$= ay - by^2$ as required.

Question 5 **(3 marks)**

Euler realised that we can use the incremental formula

$$\delta y \approx \frac{dy}{dx}\delta x$$

where δx is the step size in the values of x, to iteratively find approximate values of y.

Euler basically realised that as the x-value changes from x to $x + \delta x$, the y-value will approximately change from y to $y + \delta y$.

Now consider the differential equation

$$\frac{dy}{dx} = \frac{10}{x - 10}$$

with initial values $x = 0$ and $y = 5$.

Use Euler's method with a step size of 0.5 in the values of x to determine an approximate value of y when $x = 1.5$.

1st iteration: $y \approx 5 + \frac{10}{-10}(0.5) = 4.5$ ✓

2nd iteration: $y \approx 4.5 + \frac{10}{-9.5}(0.5) \approx 3.97$ ✓

3rd iteration: $y \approx 3.97 + \frac{10}{-9}(0.5) \approx 3.42$ ✓

Question 6 **(5 marks)**

A body moves such that its displacement from an origin O at time t seconds is x metres, where $x = 5\cos(3t) - 2\sin(3t)$.

The displacement x can be written as $x = a\sin(3t + b)$, $a > 0$.

(a) Calculate the value of a. (3 marks)

$a(\sin(3t)\cos b + \cos(3t)\sin b) = 5\cos(3t) - 2\sin(3t)$ ✓

$\therefore \begin{cases} a\cos b = -2 \\ a\sin b = 5 \end{cases}$

$\therefore \left(\frac{-2}{a}\right)^2 + \left(\frac{5}{a}\right)^2 = 1$ ✓

$\therefore 4 + 25 = a^2$

$\therefore a = \sqrt{29}$ ✓

(b) Determine a possible value of b. (2 marks)

$\cos^{-1}\left(\frac{-2}{\sqrt{29}}\right) \approx 1.95$

✓ method

✓ correct answer

Question 7 **(12 marks)**

An object on the surface of a liquid is released at time $t = 0$ and immediately sinks. Let x be the displacement in metres in a downward direction from the surface at time t seconds.

The equation of motion is given by

$$\frac{dv}{dt} = 10 - \frac{v^2}{40}$$

where v is the velocity of the object.

(a) (i) Show that, for $a > t$, $\frac{d}{dt}\left(\ln\left(\frac{a+t}{a-t}\right)\right) = \frac{2a}{a^2 - t^2}$ (3 marks)

$$\left(\ln\left(\frac{a+t}{a-t}\right)\right)' = \left(\frac{a-t}{a+t}\right)\left(\frac{1(a-t)-(a+t)(-1)}{(a-t)^2}\right)$$

✓ chain rule
✓ Quotient rule

$$= \left(\frac{a-t}{a+t}\right)\left(\frac{2a}{(a-t)^2}\right)$$

✓ correct simplification

$$= \frac{2a}{a^2-t^2} \quad \text{as required.}$$

(ii) Hence, or otherwise, show that $v = \frac{20(e^t - 1)}{e^t + 1}$ (4 marks)

$$\frac{dv}{10 - \frac{v^2}{40}} = dt$$

✓ Separate variables

$$\int \frac{40\,dv}{400 - v^2} = \int dt$$

$$\ln\left(\frac{20+v}{20-v}\right) = t + c$$

✓ use Part (i)

$$\frac{20+v}{20-v} = e^{t+c}$$

when $t=0, v=0$: $1 = e^c$ $\therefore$ $\frac{20+v}{20-v} = e^t$

✓ use $t=0$, $v=0$

$$20 + v = 20e^t - ve^t$$

$$v(1+e^t) = 20(e^t - 1)$$

✓ correct solving for v

$$v = \frac{20(e^t-1)}{e^t+1} \quad \text{as required}$$

(b) Use $\frac{dv}{dt} = v\frac{dv}{dx}$ to show that $x = 20\ln\left(\frac{400}{400 - v^2}\right)$ (3 marks)

$$10 - \frac{v^2}{40} = v\frac{dv}{dx}$$

✓ $\frac{dv}{dt} = 10 - \frac{v^2}{40}$

$$\frac{400 - v^2}{40} = v\frac{dv}{dx}$$

$$\int dx = \int \frac{40v}{400 - v^2}\,dv$$

$$x + c = -20\ln(400 - v^2)$$

when $x = 0, v = 0$: $c = -20\ln(400)$

✓ use $x=0$, $v=0$

$$\therefore\ x = 20\ln(400) - 20\ln(400 - v^2)$$

✓ correct simplification

$$= 20\ln\left(\frac{400}{400 - v^2}\right)$$ as required.

(c) How far does the object sink in the first 4 seconds? (2 marks)

when $t = 4$, $v = \frac{20(e^4 - 1)}{e^4 + 1}$ ✓

For that value of v we find that

$$x = 20\ln\left(\frac{400}{400 - v^2}\right)$$

$$\approx 53$$ ✓

$\therefore$ About 53 metres.

Question 8 **(5 marks)**

A student carried out a simulation to select a random sample of 100 packets of nails and to record the mean length of the 40 nails in each packet. The nails were assumed to have a mean length of 25 mm and a standard deviation of 2.5 mm.

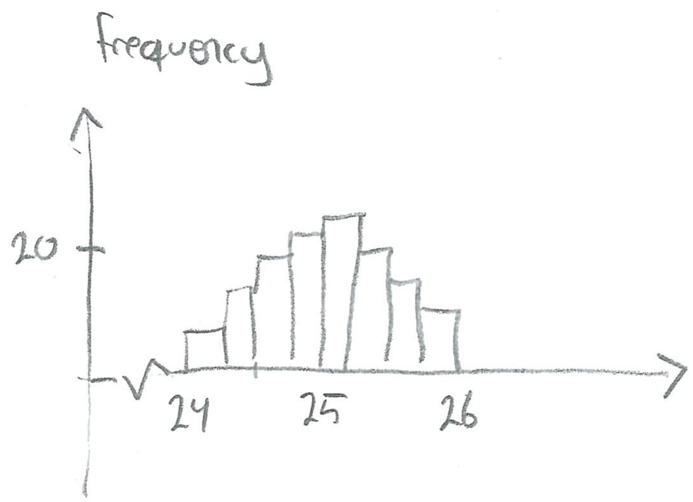
(a) Describe the expected features of a frequency graph showing the distribution of the 100 sample means. (3 marks)

Distribution of sample means would be normal. ✓

Distribution would be centred on $\bar{x} = 25$ mm ✓

Distribution would have standard deviation of $\frac{2.5}{\sqrt{40}} \approx 0.395$ mm ✓

(b) Sketch a possible frequency graph with the features described in part (a). (2 marks)



✓ scale

✓ shape

End of questions